Implicit Incompressible Porous Flow using SPH: Supplementary Material

Timna Böttcher, Lukas Westhofen, Stefan Rhys Jeske, Jan Bender

Visual Computing Institute, RWTH Aachen University, Germany

1 Experiment Parameters and Performance

Table 1 provides the simulation parameters used for each of the scenarios shown in the paper, while Table 2 gives performance measurements as average values over the whole simulation.

Table 1: ***Scene Parameters.*** *Parameters used for the scenes shown in our paper. These parameters are: particle radius r, fluid phase rest density ρ_F^0, fluid viscosity μ^{vis}, porous solid rest density $\bar{\rho}_S^0 = (1-\phi)\rho_S^0$, porosity ϕ, Young's modulus E, Poisson's ratio ν, capillary coefficient $C^{cap,0}$, capillary falloff factor η_{cap}, and porous viscosity coefficient μ^{por}. A "-" corresponds to the parameter being varied in the examples, with the concrete values already given in the paper. Values with a "*" only apply to the variants using our method, as these parameters do not exist for the variant with the method by Ren et al. [2021]. For the SIGGRAPH letters example, the solid particles are fixed in place and their density and Young's modulus E are chosen to prevent any predicted movement in the strongly coupled solver.*

Scene		Fluid		Solid				Porous		
	r [cm]	ρ_F^0 [kg m^{-3}]	μ^{vis} [Pa s]	$\bar{\rho}_S^0$ [kg m^{-3}]	ϕ [1]	E [kPa]	ν [1]	$C^{\text{cap},0}$ [N m^{-1}]	η_{cap} [1]	μ^{por} [Pa s]
Sponge (Fig. 1, 12)	1.5	1000	50.0	1000	0.6	250	0.49	1000	0.7	0.5
Porosity (Fig. 3, 9)	1.0	1000	20.0	2000	-	250	0.33	0	0.0	5.0
Capillary Action (Fig. 4, 11)	1.0	1000	20.0	1000	0.6	250	0.33	-	-	1.0
Wall (Fig. 5)	1.0	1000	20.0	3000	0.6	750	0.33	0	0.0	-
Comparison (Fig. 6)	1.0	1000	20.0	1300	0.6	250	0.33	1000*	1.0*	-*
Dam (Fig. 7)	1.0	1000	20.0	3000	0.4	1000	0.33	0*	0.0*	5.0*
Buoyancy (Fig. 8)	1.0	1000	20.0	-	-	750	0.33	0	0.0	1.0
Seepage (Fig. 10)	1.0	1000	20.0	2000	0.5	500	0.33	-	0.7	-
Fusilli (Fig. 13)	0.75	1000	20.0	1500	0.8	100000	0.4	2400	0.88	1.0
Propeller (Fig. 14)	1.0	1000	20.0	150	0.55	750	0.33	75	0.8	1.0
SIGGRAPH (Fig. 15)	0.75	1000	20.0	50000	0.75	10000000	0.49	2000	0.7	0.5

Table 2: ***Performance Measurements.*** *The simulation step includes computing the forces, as well as other procedures like neighborhood search and density computations. The solver for the non-pressure forces has a tolerance of 0.0001 for the relative residual error. The pressure solver is applied until the average density error is less than 0.05% of the rest density. The time step size of the sponge and propeller scenes are averaged due to the use of the CFL condition (all others are kept constant) and the iterations of the non-pressure forces in the examples using the method by Ren et al. [2021] only measure those of the elasticity solver.*

Scene	Num. Fluid Particles	Num. Solid Particles	Time step size [ms]	Sim. Step [ms]	Non-Pressure Forces Time [ms]	Non-Pressure Forces Iterations	Pressure Forces Time [ms]	Pressure Forces Iterations
Sponge (Fig. 1, 12)	61.5K	21.3K	0.37	43.6	7.1	0.6	24.8	2.2
Porosity (Fig. 3, 9)	110.6K	110.6K	1.00					
$\phi = 0.4$				210.6	55.4	7.2	127.4	4.5
$\phi = 0.6$				204.3	59.2	7.4	115.5	4.3
$\phi = 0.8$				245.2	64.3	7.2	148.8	4.5
Capillary Action (Fig. 4,11)	84.9K	104.7K	1.00					
$C^{\text{cap},0} = 0\text{N}\,\text{m}^{-1}, \eta_{\text{cap}} = 0.0$				156.2	48.2	4.9	75.0	2.0
$C^{\text{cap},0} = 2500\text{N}\,\text{m}^{-1}, \eta_{\text{cap}} = 0.7$				245.1	56.9	5.8	152.8	4.8
$C^{\text{cap},0} = 500\text{N}\,\text{m}^{-1}, \eta_{\text{cap}} = 1.0$				193.2	53.5	5.8	105.0	3.3
$C^{\text{cap},0} = 500\text{N}\,\text{m}^{-1}, \eta_{\text{cap}} = 0.5$				187.3	51.1	4.9	101.3	3.0
Comparison (Fig. 6)	84.9K	104.0K	1.00					
Ren et al. [2021] Slow				125.8	40.8	8.7	54.2	7.6
Ours Slow				194.1	60.8	8.0	100.3	2.6
Ren et al. [2021] Fast				186.5	42.7	8.6	111.3	16.3
Ours Fast				187.4	46.7	4.7	107.2	3.3
Dam (Fig. 7)	32.0K	15.5K	1.00					
Ren et al. [2021]				32.5	10.1	14.0	8.1	3.4
Ren et al. [2021] with gravity				41.0	10.3	14.0	15.5	7.0
Ours				38.3	15.8	10.2	13.3	2.3
Buoyancy (Fig. 8)	242.0K	18.4K	1.00					
$\rho_{\text{Solid}} = 888.9\text{kg}\,\text{m}^{-3}$				407.4	83.3	16.5	282.8	5.0
$\rho_{\text{Solid}} = 1142.9\text{kg}\,\text{m}^{-3}$				403.1	85.5	16.2	274.2	4.9
$\rho_{\text{Solid}} = 1555.6\text{kg}\,\text{m}^{-3}$				400.1	68.4	12.7	287.1	4.9
Seepage (Fig. 10)	110.6K	110.6K	1.00					
$\mu^{\text{por}} = 10\text{Pa}\,\text{s}$				304.8	71.4	6.7	192.6	5.0
$\mu^{\text{por}} = 50\text{Pa}\,\text{s}$				334.7	72.4	6.8	221.1	5.3
$C^{\text{cap},0} = 2500\text{N}\,\text{m}^{-1}$				373.7	75.6	7.1	256.2	6.0
Fusilli (Fig. 13)	1000.0K	78.7K	1.00	949.0	119.5	13.6	723.1	4.9
Propeller (Fig. 14)	875.0K	16.5K	0.53	1218.8	292.8	14.6	763.1	4.4
SIGGRAPH (Fig. 15)	425.3K	240.4K	0.50	565.0	42.2	0.2	453.6	5.5

2 Algorithm

We employ the typical SPH operator splitting method, by first solving all the non-pressure forces followed by pressure forces that correct the velocity field to ensure incompressibility. For typical SPH solvers, this further allows the use of different specialized solvers for each force (see Algorithm 1), such that the fluid viscosity solver does not generally depend on the solid elasticity solver and vice versa.

We now assume that the force formulations depend linearly on the new velocity field $\mathbf{v}^*$ for both fluid viscosity and solid elasticity. For solid elasticity, this results in a linear system for the solid particle velocities $\mathbf{v}_S$ of the form:

$$\mathbf{M}_S \mathbf{v}_S^* = \mathbf{M}_S \mathbf{v}_S^n + \Delta t \mathbf{f}^{\text{elast}}, \tag{1}$$

where $\mathbf{M}_S$ is the mass matrix. Since $\mathbf{f}^{\text{elast}}$ is linear in the velocities, we can write it as $\mathbf{f}^{\text{elast}} = \mathbf{A}^{\text{elast}} \mathbf{v}_S^* + \mathbf{b}_S^{\text{elast},n}$, where $\mathbf{b}_S^{\text{elast},n}$ only depends on known terms. We can then write the linear system as:

$$\begin{aligned} \mathbf{M}_S \mathbf{v}_S^* &= \mathbf{M}_S \mathbf{v}_S^n + \Delta t \left(\mathbf{A}^{\text{elast}} \mathbf{v}_S^* + \mathbf{b}^{\text{elast},n} \right) \\ \iff (\mathbf{M}_S - \Delta t \mathbf{A}^{\text{elast}}) \mathbf{v}_S^* &= \mathbf{M}_S \mathbf{v}_S^n + \Delta t \mathbf{b}_S^{\text{elast},n}. \end{aligned} \tag{2}$$

The right-hand side includes the current velocities $\mathbf{v}^n$ as well as elasticity forces corresponding to already present deformations (stored in $\mathbf{b}^{\text{elast},n}$), while the left-hand side captures how the elastic forces change based on the new velocities.

For fluid viscosity we instead solve the system for the fluid particle velocities $\mathbf{v}_F$ based on the force $\mathbf{f}^{\text{vis}} = \mathbf{A}^{\text{vis}} \mathbf{v}_F^*$:

$$(\mathbf{M}_F - \Delta t \mathbf{A}^{\text{vis}}) \mathbf{v}_F^* = \mathbf{M}_F \mathbf{v}_F^n. \tag{3}$$

Since viscous forces only depend on the velocity field and not on any current deformation, the corresponding term on the right-hand side is missing when compared with the elastic system ($\mathbf{b}_F^{\text{vis},n} = \mathbf{0}$).

We can now solve viscosity and elasticity at the same time, by combining the two linear systems. This will result in a system that is defined in blocks:

$$\left(\mathbf{M} - \Delta t \begin{pmatrix} \mathbf{A}^{\text{vis}} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{\text{elast}} \end{pmatrix} \right) \begin{pmatrix} \mathbf{v}_F^* \\ \mathbf{v}_S^* \end{pmatrix} = \mathbf{M} \begin{pmatrix} \mathbf{v}_F^n \\ \mathbf{v}_S^n \end{pmatrix} + \Delta t \begin{pmatrix} \mathbf{0} \\ \mathbf{b}^{\text{elast},n} \end{pmatrix}. \tag{4}$$

This mass matrix $\mathbf{M}$ is the combined mass matrix of fluid and solid particles. The upper right and lower left blocks of the system matrix correspond to coupling terms between the solid and fluid phase, which are usually not included in the non-pressure forces. Our additional coupling forces now fill these blocks with a coupling matrix $\mathbf{A}^{\text{por}}$ and add the RHS $\Delta t \mathbf{b}^{\text{por},n}$:

$$\left(\mathbf{M} - \Delta t \begin{pmatrix} \mathbf{A}^{\text{vis}} + \mathbf{A}_F^{\text{por}} & \mathbf{A}^{\text{por}} \\ (\mathbf{A}^{\text{por}})^T & \mathbf{A}^{\text{elast}} + \mathbf{A}_S^{\text{por}} \end{pmatrix} \right) \begin{pmatrix} \mathbf{v}_F^* \\ \mathbf{v}_S^* \end{pmatrix} = \mathbf{M} \begin{pmatrix} \mathbf{v}_F^n \\ \mathbf{v}_S^n \end{pmatrix} + \Delta t \begin{pmatrix} \mathbf{0} \\ \mathbf{b}^{\text{elast},n} \end{pmatrix} + \Delta t \mathbf{b}^{\text{por},n}. \tag{5}$$

Note that the matrices $\mathbf{A}_F^{\text{por}}$ and $\mathbf{A}_S^{\text{por}}$ are block-diagonal matrices, that correspond to the negative sum of all other entries from the coupling terms: $\mathbf{A}_{ii}^{\text{por}} = -\sum_j \mathbf{A}_{ij}^{\text{por}}$. Algorithm 2 illustrates how we solve a time step using this strongly-coupled system with a matrix-free conjugate gradient method, where the main terms our method adds to the standard SPH pipeline are highlighted in blue.

The right-hand side $\mathbf{b}^{\text{por},n}$ is the sum of $\mathbf{b}^{\text{cap},n}$ and $\mathbf{b}^{\text{buo},n}$, while $\mathbf{A}^{\text{por}}$ is the sum of the coupling matrices resulting from capillary action ($\mathbf{A}^{\text{cap}}$) and from porous drag ($\mathbf{A}^{\text{drag}}$). Using the linear forces given in our paper, the entries of these matrices are:

$$\mathbf{A}_{is}^{\text{cap}} = \Delta t C^{\text{cap}} (S_s^n) \frac{\bar{m}_{is}}{\bar{\rho}_{is}^n} W_{is}^n \mathbf{I}, \quad \mathbf{A}_{is}^{\text{drag}} = -2(d+2)\mu^{\text{por}} \frac{m_i m_s}{\rho_i^n \rho_s^n} \frac{\nabla W_{is}^n (\mathbf{x}_{is}^n)^T}{\|\mathbf{x}_{is}^n\|^2 + 0.01h^2} \tag{6}$$

Algorithm 1: Standard SPH Timestep

- **forall** *particles* **do**
  - Compute density ρ
- **Non-Pressure Forces**
  - **forall** *solid particles* s **do**
    - Compute elastic force $\mathbf{f}_s^{\mathrm{elast}}$
    - Update velocity $\mathbf{v}_s^*$
  - **forall** *fluid particles* i **do**
    - Compute viscosity force $\mathbf{f}_i^{\mathrm{vis}}$
    - Update velocity $\mathbf{v}_i^*$
- **Pressure Forces**
  - **forall** *particles* **do**
    - Predict density change ρ^*
  - **forall** *particles* **do**
    - Compute pressure force $\mathbf{f}^{\mathrm{press}}$
- **Integrate**
  - **forall** *particles* **do**
    - Update velocity $\mathbf{v}$
    - Update position $\mathbf{x}$

Algorithm 2: Porous SPH Timestep

- **forall** *particles* **do**
  - Compute density ρ, $\hat{\rho}$
- **Non-Pressure Forces**
  - **Compute RHS b**
    - **forall** *solid particles* s **do**
      - Compute saturation S_s
      - Compute elastic force $\mathbf{b}_s^{\mathrm{elast},n}$
    - **forall** *particles* **do**
      - Compute porous force $\mathbf{b}^{\mathrm{por},n}$
    - Assemble **b**
  - Solve Eq. (5) using matrix-free CG
- **Pressure Forces**
  - **forall** *solid particles* s, *fluid particles* i **do**
    - Predict density change ρ_s^*, $\hat{\rho}_i^*$
  - **forall** *particles* **do**
    - Compute pressure force $\mathbf{f}^{\mathrm{press}}$
- **Integrate**
  - **forall** *particles* **do**
    - Update velocity $\mathbf{v}$
    - Update position $\mathbf{x}$

3 Porosity and Density

In this section, we provide the detailed derivation of the density constraints we use to allow fluid and solid particle overlapping while still adhering to incompressibility. Here, we need to take special care of all phases present in the system (e.g., solid, water and air) to account for how much space they occupy in a given volume.

Anderson and Jackson [1967] have shown that porous flow can be modeled using volume averaging methods, where phase quantities are approximated using weighted volume integrals over the local domain Ω. Given a non-negative normalized function g, i.e. $g(\mathbf{x}) \geq 0$ and $\int_\Omega g(\mathbf{x})d\mathbf{x} = 1$, volume averaged quantities f can be computed based on their point values f':

$$f(\mathbf{x}) = \int_\Omega f'(\mathbf{y})g(\mathbf{x}-\mathbf{y})d\mathbf{y}. \tag{7}$$

In the context of porous flow, this can be used to define the porosity ϕ as the volume integral over the domain Ω_F filled by fluids:

$$\phi(\mathbf{x}) = \int_{\Omega_F} g(\mathbf{x}-\mathbf{y})d\mathbf{y} = \int_\Omega g(\mathbf{x})d\mathbf{x} - \int_{\Omega_S} g(\mathbf{x}-\mathbf{y})d\mathbf{y} = 1 - \int_{\Omega_S} g(\mathbf{x}-\mathbf{y})d\mathbf{y}, \tag{8}$$

where Ω_S is the domain occupied by the solid phase. Basically, ϕ is the volume averaged version of an indicator function ϕ', which is 1 inside the fluid domain and 0 inside the solid domain. Note that here, fluid does not just refer to the wetting fluid but all fluids present in the system, e.g. both water and air in typical porous flow scenarios.

The porosity can be used to compute volume averaged quantities from local quantities, e.g., the volume averaged solid density ρ_s based on the local point densities ρ'_s is then given by:

$$\int_\Omega \rho'_s(\mathbf{y})g(\mathbf{x}-\mathbf{y})d\mathbf{y} = \int_{\Omega_S} \rho'_s(\mathbf{y})g(\mathbf{x}-\mathbf{y})d\mathbf{y} = (1-\phi(\mathbf{x}))\,\hat{\rho}_s =: \rho_s, \tag{9}$$

where the integral can be reduced to just the solid domain since the solid density ρ'_s is non-zero only at points occupied by the solid phase. The solid phase density $\hat{\rho}$ is an averaged version of the point density ρ'_s, which is constant for incompressible solids. Therefore, we require that $\hat{\rho}^0_s = \hat{\rho}_s = \rho'_s$, where $\hat{\rho}^0_s$ is the solid phase rest density.

Similarly, we have for fluids:

$$\int_\Omega \rho'_f(\mathbf{y})g(\mathbf{x}-\mathbf{y})d\mathbf{y} = \int_{\Omega_F} \rho'_f(\mathbf{y})g(\mathbf{x}-\mathbf{y})d\mathbf{y} = \phi(\mathbf{x})\hat{\rho}_f(\mathbf{x}) =: \rho_f. \tag{10}$$

Note that here, fluids still encompass all fluids present in the system. If only the density of fluid phase w (e.g., the water density without the air) is of interest we can instead define:

$$\phi_w(\mathbf{x}) = \int_{\Omega_W} g(\mathbf{x}-\mathbf{y})d\mathbf{y} = 1 - \int_{\Omega_S} g(\mathbf{x}-\mathbf{y})d\mathbf{y} - \int_{\Omega_F/\Omega_W} g(\mathbf{x}-\mathbf{y})d\mathbf{y}, \tag{11}$$

$$\phi_w(\mathbf{x})\hat{\rho}_w(\mathbf{x}) = \int_{\Omega_W} \rho'_w(\mathbf{x})g(\mathbf{x}-\mathbf{y})d\mathbf{y}. \tag{12}$$

For the incompressible fluid w, the phase density $\hat{\rho}_w$ is constant, while the volume average changes based on the local porosity $\phi_w(\mathbf{x})$. Given a known fluid phase density $\hat{\rho}_w$ we can therefore write:

$$\begin{aligned} \phi_w(\mathbf{x})\hat{\rho}_w &= \left(\int_{\Omega_W} g(\mathbf{x}-\mathbf{y})d\mathbf{y}\right)\hat{\rho}_w = \left(1 - \int_{\Omega_S} g(\mathbf{x}-\mathbf{y})d\mathbf{y} - \int_{\Omega_F/\Omega_W} g(\mathbf{x}-\mathbf{y})d\mathbf{y}\right)\hat{\rho}_w \\ \iff \left(\int_{\Omega_W} g(\mathbf{x}-\mathbf{y})d\mathbf{y} + \int_{\Omega_S} g(\mathbf{x}-\mathbf{y})d\mathbf{y} - 1\right)\hat{\rho}_w &= -\left(\int_{\Omega_F/\Omega_W} g(\mathbf{x}-\mathbf{y})d\mathbf{y}\right)\hat{\rho}_w. \end{aligned} \tag{13}$$

Since g is non-negative at all points, the right-hand side is less than or equal to zero (based on how much air is present), therefore:

$$\begin{aligned} &\left(\int_{\Omega_W} g(\mathbf{x}-\mathbf{y})d\mathbf{y} + \int_{\Omega_S} g(\mathbf{x}-\mathbf{y})d\mathbf{y} - 1\right)\hat{\rho}_w \leq 0 \\ \iff\ &\hat{\rho}_w \left(\int_{\Omega_W} g(\mathbf{x}-\mathbf{y})d\mathbf{y} + \int_{\Omega_S} g(\mathbf{x}-\mathbf{y})d\mathbf{y}\right) \leq \hat{\rho}_w. \end{aligned} \tag{14}$$

This condition can be readily approximated using SPH, where the SPH kernel W takes on the role of g. Given a local phase density $\hat{\rho}$, a particle with mass m will occupy a volume of $\hat{V} = \frac{m}{\hat{\rho}}$. Therefore, the integral over the solid domain can be approximated as:

$$\int_{\Omega_S} g(\mathbf{x}_i-\mathbf{y})d\mathbf{y} \approx \sum_{t\in\mathcal{N}_i^S} \hat{V}_t W_{it}. \tag{15}$$

The same holds for the integral over the fluid domain using nearby fluid particles, such that we get:

$$\hat{\rho}_w \left(\int_{\Omega_W} g(\mathbf{x}-\mathbf{y})d\mathbf{y} + \int_{\Omega_S} g(\mathbf{x}-\mathbf{y})d\mathbf{y}\right) \approx \hat{\rho}_w \left(\sum_{j\in\mathcal{N}_i^F} \hat{V}_j W_{ij} + \sum_{t\in\mathcal{N}_i^S} \hat{V}_t W_{it}\right) \leq \hat{\rho}_w. \tag{16}$$

Since both fluid and solid phase are incompressible ($\hat{\rho} = \text{const}$) and the particles have constant mass m, the covered volume $\hat{V}$ is also constant. For fluid particles, we therefore have $\hat{V}_i = V_i^0$, where V_i^0 is the particles initial or sampling volume.

In our method, the solid particles are instead sampled with a volume that does not just cover the solid domain, but also the pore space. We therefore need to correct the solid particle volumes. Here, we can use the definition of the rest porosity ϕ^0, which is a constant material parameter. Since we sample the porous solid in its rest state, we know that the solid phase only covers a fraction of $1-\phi^0$ of each particle volume V_s^0. The constant covered volume is therefore $\hat{V}_s = (1-\phi^0)V_s^0$ and we have:

$$\sum_{t\in\mathcal{N}_i^S} \hat{V}_t W_{it} = (1-\phi^0) \sum_{t\in\mathcal{N}_i^S} V_t^0 W_{it}, \tag{17}$$

from which follows the condition:

$$\hat{\rho}_w \left(\sum_{j\in\mathcal{N}_i^F} V_j^0 W_{ij} + (1-\phi^0) \sum_{t\in\mathcal{N}_i^S} V_t^0 W_{it}\right) \leq \hat{\rho}_w. \tag{18}$$

This condition has the same structure as the density constraint used in SPH pressure solvers, which we therefore employ to ensure fluid incompressibility.

For solid particles, we use:

$$(1-\phi(\mathbf{x}))\,\hat{\rho}_s = \left(\int_{\Omega_S} g(\mathbf{x}-\mathbf{y})d\mathbf{y}\right)\hat{\rho}_s \leq \hat{\rho}_s, \tag{19}$$

where the inequality follows from g being normalized and $\Omega_S \subseteq \Omega$. Using the same arguments as for the fluid density, the integral over the solid domain can be approximated using an SPH sum:

$$\hat{\rho}_s \left((1-\phi^0) \sum_{t\in\mathcal{N}_s^S} V_t^0 W_{st}\right) \leq \hat{\rho}_s. \tag{20}$$

From a visual perspective, this condition is violated if the solid particle overlap exceeds the pore space, which is not permitted due to the assumption that the solid phase is incompressible.

If we instead assume that the porous object itself is incompressible, i.e., the pores do not change in size and the local porosity $\phi(\mathbf{x}) = \phi^0$ is also constant, we can further see that:

$$\begin{aligned} 1 - \phi(\mathbf{x}) = \int_{\Omega_S} g(\mathbf{x} - \mathbf{y}) d\mathbf{y} \approx (1 - \phi^0) \sum_{t \in \mathcal{N}_s^S} V_t^0 W_{st} \\ \Rightarrow \sum_{t \in \mathcal{N}_s^S} V_t^0 W_{st} \approx 1. \end{aligned} \tag{21}$$

This will lead to issues at the surface of the porous solid due to particle neighborhood deficiency (i.e., the particles t do not cover the whole space around the current particle s at position $\mathbf{x}$, such that the volume integral is underestimated). We therefore soften this equality and instead require that the porosity may not decrease below the rest porosity ($\phi(\mathbf{x}) \geq \phi^0$), such that solid particles are not allowed to move closer together, as that would correspond to volume loss. We then derive the constraint:

$$\begin{aligned} 1 - \phi(\mathbf{x}) \approx (1 - \phi^0) \sum_{t \in \mathcal{N}_s^S} V_t^0 W_{st} \leq 1 - \phi^0 \\ \Rightarrow \hat{\rho}_s \left((1 - \phi^0) \sum_{t \in \mathcal{N}_s^S} V_t^0 W_{st} \right) \leq \hat{\rho}_s (1 - \phi^0) = \rho_s. \end{aligned} \tag{22}$$

We therefore have two different upper bounds for the same density estimation, based on the assumption that either the porous solid is incompressible (reduction of pore space is not allowed with upper bound $\hat{\rho}_s(1 - \phi^0)$) or that only the solid phase itself is incompressible (upper bound $\hat{\rho}_s$).

4 Porous Forces and Darcy's Law

Porous interaction forces are generally added to the momentum equations as special external forces, acting on the surfaces of the phase continuum. The momentum equation for a fluid with negligible viscosity is then:

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathbf{f}^{\text{por}} + \mathbf{g}. \tag{23}$$

While the specific form of $\mathbf{f}^{por}$ is unknown, one possible form is a linear drag force based on the relative fluid velocity $\mathbf{v}_{fs} = \mathbf{v}_f - \mathbf{v}_s$, with local solid velocity $\mathbf{v}_s$ and fluid velocity $\mathbf{v}_f$:

$$\mathbf{f}^{\text{por}} = -K\mathbf{v}_{fs}. \tag{24}$$

Here, $K = \frac{\phi\mu}{k}$ is assumed to be proportional to the fluid viscosity μ and the solid permeability k, such that higher viscosity or lower permeability increase the drag force. The porosity ϕ is included to convert the velocity into the volume flux. If we then assume stationary conditions, i.e. $\frac{D\mathbf{v}}{Dt} = \mathbf{0}$, we get:

$$\begin{aligned} \mathbf{0} &= -\frac{1}{\rho}\nabla p - \frac{1}{\rho}K\mathbf{v}_{fs} + \mathbf{g} \\ \iff \frac{1}{\rho}K\mathbf{v}_{fs} &= -\frac{1}{\rho}\nabla p + \mathbf{g} \\ \iff \mathbf{v}_{fs} &= K^{-1}(-\nabla p + \rho\mathbf{g}) = -\frac{k}{\phi\mu}(\nabla p - \rho\mathbf{g}). \end{aligned} \tag{25}$$

This is Darcy's law, where gravity can sometimes be neglected depending on the pressure forces and the simulation scenario.

In many computer graphics applications we cannot assume stationary conditions, since we usually aim for dynamic scenes. Additionally, computing a physically accurate pressure field for Darcy's law is challenging, especially if fluid incompressibility and capillary action have to be considered. Therefore, the pressure field is often approximated using heuristics based on the local fluid saturation (e.g. Lenaerts et al. [2008] or Ren et al. [2021]). Our model avoids these issues by not using Darcy's law for the relative fluid velocity, as we instead consider the forces directly.

References

T. B. Anderson and Roy Jackson. Fluid mechanical description of fluidized beds. equations of motion. *Industrial & Engineering Chemistry Fundamentals*, 6:527–539, 1967.

Toon Lenaerts, Bart Adams, and Philip Dutré. Porous flow in particle-based fluid simulations. *ACM Transactions on Graphics*, 27 (3):1–8, August 2008. ISSN 0730-0301. doi: 10.1145/1360612.1360648.

Bo Ren, Ben Xu, and Chenfeng Li. Unified particle system for multiple-fluid flow and porous material. *ACM Transactions on Graphics*, 40(4), July 2021. ISSN 0730-0301. doi: 10.1145/3450626.3459764.