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Fast Corotated FEM using Operator Splitting

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1. Gradients of the Volume Constraint

In order to compute the gradients of the volume constraint

$$C_t(\mathbf{x}) = \det(\mathbf{F}_t) - 1 = \frac{[(\mathbf{x}_1 - \mathbf{x}_0) \times (\mathbf{x}_2 - \mathbf{x}_0)]\,(\mathbf{x}_3 - \mathbf{x}_0)}{6V_t} - 1,$$

we use the fact that the scalar triple product can be cyclically permuted and the fact that the sum of the gradients is $\mathbf{0}$:

$$\begin{aligned}
\nabla_{\mathbf{x}_1} C_t &= \frac{1}{6v_t}(\mathbf{x}_{20} \times \mathbf{x}_{30}) \\
\nabla_{\mathbf{x}_2} C_t &= \frac{1}{6v_t}(\mathbf{x}_{30} \times \mathbf{x}_{10}) \\
\nabla_{\mathbf{x}_3} C_t &= \frac{1}{6v_t}(\mathbf{x}_{10} \times \mathbf{x}_{20}) \\
\nabla_{\mathbf{x}_0} C_t &= -\nabla_{\mathbf{x}_1} C_t - \nabla_{\mathbf{x}_2} C_t - \nabla_{\mathbf{x}_3} C_t,
\end{aligned}$$

where $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$.

2. Gradient and Hessian of E_{APD}

To compute the gradient and Hessian of the function E_{APD}we have to determine the derivatives of $\mathbf{R}(\boldsymbol{\omega})$ what can be done component wise by using the exponential series

$$\begin{aligned}
\frac{\partial}{\partial \omega_k}\left(\mathbf{R}_0 \exp\left([\boldsymbol{\omega}]_\times\right)\right)\Big|_{\boldsymbol{\omega}=\mathbf{0}} &= \mathbf{R}_0 \frac{\partial}{\partial \omega_k}\left(\sum_{i=0}^{\infty} \frac{1}{i!}[\boldsymbol{\omega}]_\times^i\right)\Big|_{\boldsymbol{\omega}=\mathbf{0}} \\
&= \mathbf{R}_0 [\mathbf{e}_k]_\times \\
\frac{\partial^2}{\partial \omega_k \partial \omega_l}\left(\mathbf{R}_0 \exp\left([\boldsymbol{\omega}]_\times\right)\right)\Big|_{\boldsymbol{\omega}=\mathbf{0}} &= \mathbf{R}_0 \frac{\partial^2}{\partial \omega_k \partial \omega_l}\left(\sum_{i=0}^{\infty} \frac{1}{i!}[\boldsymbol{\omega}]_\times^i\right)\Big|_{\boldsymbol{\omega}=\mathbf{0}} \\
&= \tfrac{1}{2}\mathbf{R}_0\left([\mathbf{e}_k]_\times [\mathbf{e}_l]_\times + [\mathbf{e}_l]_\times [\mathbf{e}_k]_\times\right).
\end{aligned}$$

In the first derivative evaluated at $\boldsymbol{\omega} = \mathbf{0}$ there is only one non-vanishing term that comes from the linear term in the exponential series and in the second derivative there are only two terms left that come from the quadratic term.

These derivatives can be used to compute the gradient and Hessian of

$$E_{\text{APD}}(\mathbf{R}(\boldsymbol{\omega})) = \tfrac{1}{2} \parallel \mathbf{R} - \mathbf{A} \parallel^2 = \tfrac{1}{2}\sum_{i=1}^{3}(\mathbf{R}\mathbf{e}_i - \mathbf{A}\mathbf{e}_i)^2. \quad (1)$$

The gradient is determined as

$$\begin{aligned}
\frac{\partial}{\partial \omega_k} E_{\text{APD}} &= \sum_i \left(\frac{\partial \mathbf{R}(\omega)}{\partial \omega_k}\mathbf{e}_i\right)^T (\mathbf{R}\mathbf{e}_i - \mathbf{A}_i) \\
&= \sum_i \left(\mathbf{R}_0 [\mathbf{e}_k]_\times \mathbf{e}_i\right)^T (\mathbf{R}\mathbf{e}_i - \mathbf{A}_i).
\end{aligned}$$

During the minimization we will evaluate the gradient at $\mathbf{R} = \mathbf{R}_0$ which leads to (to simplify notation we will denote $\mathbf{A}\mathbf{e}_i$ as $\mathbf{A}_i$ and $\mathbf{R}\mathbf{e}_i$ as $\mathbf{r}_i$):

$$\begin{aligned}
\frac{\partial}{\partial \omega_1} E_{\text{APD}} &= \sum_i \left(\mathbf{R}_0 [\mathbf{e}_1]_\times \mathbf{e}_i\right)^T (\mathbf{R}\mathbf{e}_i - \mathbf{A}_i) \\
&= (\mathbf{R}_0\mathbf{e}_3)^T(\mathbf{R}_0\mathbf{e}_2 - \mathbf{A}_2) - (\mathbf{R}_0\mathbf{e}_2)^T(\mathbf{R}_0\mathbf{e}_3 - \mathbf{A}_3) \\
&= -(\mathbf{R}_0\mathbf{e}_3)^T\mathbf{A}_2 + (\mathbf{R}_0\mathbf{e}_2)^T\mathbf{A}_3 \\
&= -(\mathbf{r}_3)^T\mathbf{A}_2 + (\mathbf{r}_2)^T\mathbf{A}_3
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \omega_2} E_{\text{APD}} &= -(\mathbf{R}_0\mathbf{e}_3)^T(\mathbf{R}_0\mathbf{e}_1 - \mathbf{A}_1) + (\mathbf{R}_0\mathbf{e}_1)^T(\mathbf{R}_0\mathbf{e}_3 - \mathbf{A}_3) \\
&= (\mathbf{r}_3)^T\mathbf{A}_1 - (\mathbf{r}_1)^T\mathbf{A}_3
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \omega_3} E_{\text{APD}} &= (\mathbf{R}_0\mathbf{e}_2)^T(\mathbf{R}_0\mathbf{e}_1 - \mathbf{A}_1) - (\mathbf{R}_0\mathbf{e}_1)^T(\mathbf{R}_0\mathbf{e}_2 - \mathbf{A}_2) \\
&= -(\mathbf{r}_2)^T\mathbf{A}_1 + (\mathbf{r}_1)^T\mathbf{A}_2.
\end{aligned}$$

Further, we compute the Hessian as

$$\begin{aligned}
\frac{\partial^2}{\partial \omega_j \partial \omega_k} E_{\text{APD}} &= \sum_i \left(\frac{\partial^2 \mathbf{R}(\omega)}{\partial \omega_j \partial \omega_k}\mathbf{e}_i\right)^T (\mathbf{R}\mathbf{e}_i - \mathbf{A}_i) \\
&\quad + \sum_i \left(\frac{\partial \mathbf{R}(\omega)}{\partial \omega_k}\mathbf{e}_i\right)^T \left(\frac{\partial \mathbf{R}(\omega)}{\partial \omega_j}\mathbf{e}_i\right) \\
&= \sum_i \left(\tfrac{1}{2}\mathbf{R}_0([\mathbf{e}_j]_\times [\mathbf{e}_k]_\times + [\mathbf{e}_k]_\times [\mathbf{e}_j]_\times)\mathbf{e}_i\right)^T (\mathbf{R}\mathbf{e}_i - \mathbf{A}_i) \\
&\quad + \sum_i \left(\mathbf{R}_0 [\mathbf{e}_k]_\times \mathbf{e}_i\right)^T \left(\mathbf{R}_0 [\mathbf{e}_j]_\times \mathbf{e}_i\right).
\end{aligned}$$

This results in the elements

$$\begin{aligned}
\frac{\partial^2}{\partial\omega_1\partial\omega_1}E_{\text{APD}} &= \sum_i \left(\tfrac{1}{2}\mathbf{R}_0([\mathbf{e}_1]_\times[\mathbf{e}_1]_\times + [\mathbf{e}_1]_\times[\mathbf{e}_1]_\times)\mathbf{e}_i\right)^T(\mathbf{R}\mathbf{e}_i - \mathbf{A}_i) \\
&\quad + \sum_i \left(\mathbf{R}_0[\mathbf{e}_1]_\times\mathbf{e}_i\right)^T\left(\mathbf{R}_0[\mathbf{e}_1]_\times\mathbf{e}_i\right) \\
&= -\mathbf{r}_2^T(\mathbf{r}_2 - \mathbf{A}_2) - \mathbf{r}_3^T(\mathbf{r}_3 - \mathbf{A}_3) + 2 \\
&= \mathbf{r}_2^T\mathbf{A}_2 + \mathbf{r}_3^T\mathbf{A}_3 \\
\frac{\partial^2}{\partial\omega_2\partial\omega_2}E_{\text{APD}} &= -\mathbf{r}_1^T(\mathbf{r}_1 - \mathbf{A}_1) - \mathbf{r}_3^T(\mathbf{r}_3 - \mathbf{A}_3) + 2 \\
&= \mathbf{r}_1^T\mathbf{A}_1 + \mathbf{r}_3^T\mathbf{A}_3 \\
\frac{\partial^2}{\partial\omega_3\partial\omega_3}E_{\text{APD}} &= \mathbf{r}_1^T\mathbf{A}_1 + \mathbf{r}_2^T\mathbf{A}_2 \\
\frac{\partial^2}{\partial\omega_1\partial\omega_2}E_{\text{APD}} &= \sum_i \left(\tfrac{1}{2}\mathbf{R}_0([\mathbf{e}_1]_\times[\mathbf{e}_2]_\times + [\mathbf{e}_2]_\times[\mathbf{e}_1]_\times)\mathbf{e}_i\right)^T(\mathbf{R}\mathbf{e}_i - \mathbf{A}_i) \\
&\quad + \sum_i \left(\mathbf{R}_0[\mathbf{e}_2]_\times\mathbf{e}_i\right)^T\left(\mathbf{R}_0[\mathbf{e}_1]_\times\mathbf{e}_i\right) \\
&= \tfrac{1}{2}\mathbf{r}_2^T(\mathbf{r}_1 - \mathbf{A}_1) + \tfrac{1}{2}\mathbf{r}_1^T(\mathbf{r}_2 - \mathbf{A}_2) \\
&= -\tfrac{1}{2}\mathbf{r}_1^T\mathbf{A}_2 - \tfrac{1}{2}\mathbf{r}_2^T\mathbf{A}_1 \\
\frac{\partial^2}{\partial\omega_1\partial\omega_3}E_{\text{APD}} &= -\tfrac{1}{2}\mathbf{r}_1^T\mathbf{A}_3 - \tfrac{1}{2}\mathbf{r}_3^T\mathbf{A}_1 \\
\frac{\partial^2}{\partial\omega_2\partial\omega_3}E_{\text{APD}} &= -\tfrac{1}{2}\mathbf{r}_2^T\mathbf{A}_3 - \tfrac{1}{2}\mathbf{r}_3^T\mathbf{A}_2
\end{aligned}$$

The rest of the elements are determined by the symmetry of the Hessian. To simplify notation we define the matrix $\mathbf{Z}$ with $Z_{ij} = \mathbf{r}_i^T\mathbf{A}_j$. Combining the results from above yields

$$\frac{\partial}{\partial\omega}E_{\text{APD}} = \begin{pmatrix} Z_{23} - Z_{32} \\ Z_{31} - Z_{13} \\ Z_{12} - Z_{21} \end{pmatrix} = -2\,\text{axl}(\mathbf{Z})$$

$$\begin{aligned}
\frac{\partial^2}{\partial\omega^2}E_{\text{APD}} &= \begin{pmatrix} Z_{22}+Z_{33} & -\frac{1}{2}(Z_{12}+Z_{21}) & -\frac{1}{2}(Z_{13}+Z_{31}) \\ -\frac{1}{2}(Z_{12}+Z_{21}) & Z_{11}+Z_{33} & -\frac{1}{2}(Z_{23}+Z_{32}) \\ -\frac{1}{2}(Z_{13}+Z_{31}) & -\frac{1}{2}(Z_{23}+Z_{32}) & Z_{11}+Z_{22} \end{pmatrix} \\
&= -\mathbf{Z} + \text{tr}(\mathbf{Z})\,\mathbb{1} + \tfrac{1}{2}\mathbf{Z} - \tfrac{1}{2}\mathbf{Z}^T \\
&= \text{tr}(\mathbf{Z})\,\mathbb{1} - \text{sym}(\mathbf{Z}).
\end{aligned}$$

3. Proof of Equivalence of APD and IFE

Here, we will prove that the APD is equivalent to the invertible finite elements (IFE) approach of Irving et al. [ITF04]. This can be shown by rewriting the optimization problem as:

$$\begin{aligned}
\min_{\mathbf{R}\in SO(3)} \| \mathbf{F} - \mathbf{R} \| &= \min_{\mathbf{R}\in SO(3)} \sum_i \| \mathbf{F}_i - \mathbf{R}\mathbf{e}_i \| \\
&= \min_{\mathbf{R}\in SO(3)} \sum_i \| \mathbf{F}_i \|^2 \\
&\qquad - 2\text{tr}(\mathbf{R}^T \sum_i \mathbf{F}_i\mathbf{e}_i^T) + \sum_i \| \mathbf{e}_i \|^2 \\
&= \max_{\mathbf{R}\in SO(3)} \text{tr}(\mathbf{R}^T \sum_i \mathbf{F}_i\mathbf{e}_i^T) \\
&= \max_{\mathbf{R}\in SO(3)} \text{tr}(\mathbf{R}^T\mathbf{F}).
\end{aligned}$$

It was shown by Kanatani [Kan94] that the maximum can be found by computing the SVD $\mathbf{F} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ from which we can find the optimal rotation

$$\mathbf{R}_{\max} = \mathbf{U}\begin{pmatrix} 1 & & \\ & 1 & \\ & & \det(\mathbf{U}\mathbf{V}^T) \end{pmatrix}\mathbf{V}^T,$$

which is exactly the heuristic approach of Irving et al. [ITF04], where the minimal singular value is chosen to be negative and the corresponding singular vector gets inverted.

References

[ITF04] IRVING G., TERAN J., FEDKIW R.: Invertible Finite Elements For Robust Simulation of Large Deformation. In *ACM SIGGRAPH/Eurographics Symposium on Computer Animation* (2004), Eurographics Association, pp. 131–140. 2

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